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A Matching Theorem for Locally Stationary Random Processes

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A MATCHING THEOREM FOR LOCALLY STATIONARY RANDOM PROCESSES

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January, 1959

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Abstract

A theory is developed which shows the smoothing and aperture-limiting operations required to convert an arbitrary integrable non-negative function and an arbitrary continuous correlation function into components of a locally stationary random process. The harmonic analysis of locally stationary processes is examined and found to be free of certain complications that occur in the case of ordinary stationary processes.

Table of Contents

		Page
1.	Introduction	1
2.	Statement of the matching problem	4
3.	The matching theorem	7
4.	Conclusion	13
	Appendix	14
	References	16

1. Introduction

Recently the author introduced the notion of a locally stationary random process [1] and discussed an application of the theory to the problem of radio scattering by dielectric noise [2]. In this paper we discuss an interesting matching problem arising in the theory of locally stationary random processes. We begin by recalling the basic definition of [1]: *A zero-mean (separable) random process x(t), defined for all real t, is said to be locally stationary (in the wide sense)* if its covariance* $\Gamma(t,t') = Ex(t)\overline{x}(t')$ *can be written in the form*

$$\Gamma(t,t') = \Gamma_1[(t+t')/2]\ \Gamma_2(t-t') \quad , \tag{1}$$

where Γ_1 *is a non-negative function and* Γ_2 *is a stationary covariance (synonymously an autocorrelation function).* (The symbol E denotes the expectation or ensemble average, and the overbar denotes the complex conjugate.) Covariances of the form (1) are called *locally stationary covariances*. We shall often refer to $\Gamma_1(t)$ as the *power schedule* of x(t), since apart from the positive constant $\Gamma_2(0)$, which is in general not unity, $\Gamma_1(t)$ is just the average instantaneous power of the process, where t is thought of as time. In a sense $\Gamma_1(t)$ may be said to describe the *global* structure of x(t), since it can be inferred only by examining x(t) on the whole real line. On the other hand, the factor $\Gamma_2(t-t')$, which is invariant under time shifts, may be said to describe the *local* structure

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* We shall be concerned only with wide sense (i.e. second order) properties of random processes.

of $x(t)$, since it can be estimated from a knowledge of $x(t)$ in any sufficiently large interval.

For the considerations that follow, it is important to impose some regularity conditions on $\Gamma(t,t')$. The most natural requirements, both from a mathematical and physical point of view, are the following:

1. Continuity condition. We assume that $\Gamma(t,t')$ is continuous for all t,t'. As is well known, continuity of $\Gamma(t,t')$ everywhere is implied by continuity of $\Gamma(t,t')$ on the line $t=t'$, and is equivalent to mean square continuity of $x(t)$ (see [3], p.470). It is easy to see that continuity of $\Gamma(t,t')$ is equivalent to continuity of $\Gamma_1(\tau)$ and $\Gamma_2(\tau)$.

2. Integrability conditions. We assume that both $\int \Gamma_1(\tau)d\tau$ and $\int |\Gamma_2(\tau)|d\tau$ are finite,* which implies (but is not implied by) the existence of $\iint |\Gamma(t,t')|dtdt'$. This condition corresponds to the physically reasonable assumption that both the average total energy and the memory of the process be finite (where $\int|\Gamma_2(\tau)|d\tau$ is used as a measure of memory). We also assume that the Fourier transform of $\Gamma_1(\tau)$ is absolutely integrable.**

It is important to note that these regularity conditions exclude certain limiting cases of (1): The case of ordinary (or, as we shall sometimes say, globally) stationary random processes corresponds to setting Γ_1 equal to a positive constant, but then Γ_1 is not integrable. The case of locally stationary white noise[1] corresponds to setting $\Gamma_2(t-t')$ equal to the Kronecker delta function $\Delta(t-t')$, defined as 1 when $t=t'$ and 0 otherwise, but then Γ_2 is not continuous. The case of locally stationary white noise with infinite average noise power corresponds to setting $\Gamma_2(t-t')$ equal to the

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* When limits of integration are not explicitly written, the integral extends from $-\infty$ to ∞.

** This condition is needed for the proof of Lemma 3.

Dirac delta function $\delta(t-t')$, but then both the continuity and integrability requirements fail. Recalling from [1] that locally stationary white noise is needed for the harmonic analysis of globally stationary processes, we see that on all counts globally stationary processes represent a singular case in the present theory. Thus, globally stationary processes are in a sense more difficult to handle than locally stationary processes and require a special analytical tool (the stochastic integral), which is not otherwise necessary. This point is discussed further in the Appendix.

As discussed in [1], the construction of locally stationary covariances which satisfy the continuity and integrability conditions is non-trivial. For suppose we take Γ_1 to be a continuous covariance and then rely on the closure of the class of covariances under multiplication (see [3], p. 468) to make Γ a covariance. We are then confronted by the fact that continuous covariances which are functions of the variable $(t+t')/2$, the so-called exponentially convex covariances[4], have the canonical representation

$$\Gamma_1(\tau) = \int \exp(-\sigma\tau)p(\sigma)d\sigma \quad , \tag{2}$$

where $p(\sigma)$ is an integrable non-negative function, such that the integral (2) exists for all finite τ. (More generally, $\Gamma_1(\tau) = \int \exp(-\sigma\tau)dF(\sigma)$, where $F(\sigma)$ is a non-decreasing function of finite total variation, but the representation (2) is sufficient if we exclude the unphysical case where $F(\sigma)$ has a singular part and allow $p(\sigma)$ to contain delta function terms.) It follows that $\Gamma_1(\tau)$ must perforce become infinite either as $\tau \to -\infty$ or $\tau \to \infty$. This is even more unphysical than the assumption made in the globally stationary case that the process $x(t)$ exists forever with constant average power. Thus, if $\Gamma_1(\tau)$ is an exponentially convex covariance, it cannot be integrable. In

particular $\Gamma(t,t')$ will have no Fourier transform.

The way out of this difficulty adopted in [1] is the following: We start with a locally stationary process $x(t)$ with covariance (1), where Γ_1 is exponentially convex and Γ_2 is a continuous (but not necessarily absolutely integrable) autocorrelation function. We then go over to the new random process $y(t) = \exp(-at^2/2)x(t)$, $0 < a < \infty$, obtained by multiplying $x(t)$ by a Gaussian cutoff or tapering factor. For the covariance of $y(t)$ we find

$$(3)\quad \Gamma_y(t,t') = \Gamma_{1y}\left[(t+t')/2\right]\Gamma_{2y}(t-t') =$$

$$= \exp\left[-\frac{a}{4}(t+t')^2\right]\Gamma_1\left[(t+t')/2\right]\exp\left[-\frac{a}{4}(t-t')^2\right]\Gamma_2(t-t') \quad ,$$

which is still locally stationary, but no longer exponentially convex. We note that $\Gamma_{2y}(\tau)$ is absolutely integrable even if $\Gamma_2(\tau)$ is not, since $\exp(-a\tau^2/2)$ is integrable and $\Gamma_2(\tau)$ is bounded above by $\Gamma_2(0)$. If now the function $p(\sigma)$ figuring in (2) is such that the integral $\int \exp(-a\tau^2)\Gamma_1(\tau)d\tau$ is finite, then $\Gamma_{1y}(\tau)$ is integrable, and the new locally stationary process $y(t)$ obeys the continuity and integrability conditions. Let us denote by Λ the class of all locally stationary processes with covariances constructed in this way. Surprisingly enough, as we shall see below, despite its seemingly rather special nature, Λ is a very large class, which appears to be quite ample for all practical purposes.

2. Statement of the matching problem

Apparently, a basic problem in the present theory is to ascertain which noise processes are either exactly locally stationary or can be approximated in some appropriate fashion by locally stationary processes. Qualitatively speaking, if a process is such that its average instantaneous power is slowly varying with respect to its memory or correlation time

(suitably defined), then it should be possible to approximate it closely by a locally stationary process. To justify this remark, we recall first that a function $\Gamma(t,t')$, defined for all real t and t', is a covariance if and only if

$$(4) \qquad \sum_{t,t'\epsilon R_n} \Gamma(t,t')h(t)\bar{h}(t') \geq 0$$

where t and t' separately range over the points of R_n, a set of any n real numbers, and the integer n and the complex-valued function h(t) are arbitrary (see[3],p.466). Specializing (4) to the case of locally stationary covariances, and choosing $\Gamma_2(t-t')$ to be the Kronecker delta function $\Delta(t-t')$, we see immediately that the product $\Gamma_1\left[(t+t')/2\right]\Delta(t-t')$ is of non-negative definite type, and is hence a locally stationary covariance if Γ_1 is <u>any</u> non-negative function. (This is the case of locally stationary white noise.) Since any non-negative function is,in a manner of speaking, slowly varying with respect to the infinitely narrow autocorrelation function $\Delta(t-t')$, this supports the heuristic remark made above. On the other hand, if the autocorrelation function Γ_2 has finite width, we must impose further requirements on Γ_1 in addition to its being non-negative.* For example, we can easily show that in this case $\Gamma_1(t)$ cannot be allowed to vanish for any finite t (except in the trivial case where $\Gamma_1(t)$ vanishes identically). For suppose that $\Gamma_1(t)$ vanishes for some finite t and that Γ_2 has finite width, i.e. is not proportional to $\Delta(t-t')$. Then by the continuity of Γ_1 and Γ_2 there exist two points t and t' such that $\Gamma_1(t)=0$, $\Gamma_1\left[(t+t')/2\right]\neq 0$, and $\Gamma_2(t-t')\neq 0$. Then choosing R_n in (4) to consist of these point only, and specializing to real h(t), we find that the expression

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* From now on we assume that the continuity and integrability conditions, waived in the preceding example, are in force.

(5) $$2\Gamma_1\left[(t+t')/2\right]\,\Gamma_2(t-t')h(t)h(t') + \Gamma_1(t')\,\Gamma_2(0)h^2(t')$$

must be non-negative for <u>all</u> choices of $h(t)$ and $h(t')$ if $\Gamma_1\left[(t+t')/2\right]\Gamma_2(t-t')$ is to be a covariance. However, since the coefficient of $h(t)h(t')$ in (5) is non-zero, (5) can obviously be made negative by a suitable choice of $h(t)$ and $h(t')$. This establishes by contradiction that $\Gamma_1(t)$ must not vanish for finite t.

Another example indicates further the care needed in choosing Γ_1 and Γ_2 if their product (1) is to be a covariance. Consider the locally stationary covariance

(6) $$\exp(-at^2/2)\exp(-at'^2/2) = \exp\left[-\frac{a}{4}(t+t')^2\right]\exp\left[-\frac{a}{4}(t-t')^2\right]\ ,\ 0 < a < \infty,$$

already discussed in connection with (3)..Since (6) remains a covariance when multiplied by the function $\exp\left[-\frac{a'}{4}(t-t')^2\right]$, $0 < a' < \infty$, it is apparent that

(7) $$\exp\left[-\frac{a}{4}(t+t')^2\right]\exp\left[-\frac{b}{4}(t-t')^2\right]$$

is a locally stationary covariance if $b > a$. Thus, if we reduce the width of the second factor in (7) while holding the width of the first factor constant, the product remains a locally stationary covariance. On the other hand, if we reduce the width of the first factor in (7) while holding the width of the second factor constant, the product ceases to be a covariance. To see this, replace $\Gamma(t,t')$ in (4) by (7), choose R_n to consist of two distinct points $-\tau$ and τ, and set $h(-\tau) = -h(\tau) = 1$. Then (4) reduces to $2\left[\exp(-a\tau^2) - \exp(-b\tau^2)\right]$, which is negative for $a > b$. Thus, if $a > b$, the product (7) is not of non-negative definite type, and hence is not a covariance.

These examples show that in general we cannot expect to take a proposed

power schedule γ_1 and a proposed autocorrelation function γ_2 and expect that the product $\gamma_1\left[(t+t')/2\right]\gamma_2(t-t')$ will be a locally stationary covariance. In particular, except for some very special choices of γ_1 and γ_2, we expect that some preliminary operations will have to be performed on γ_1 and γ_2 before they can " match each other". More specifically, we hope to find an operation carrying γ_1 into a new non-negative function Γ_1 and an operation carrying γ_2 into a new autocorrelation function Γ_2, and such that the product $\Gamma_1\left[(t+t')/2\right]\Gamma_2(t-t')$ is a locally stationary covariance, even though the product $\gamma_1\left[(t+t')/2\right]\gamma_2(t-t')$ is not. We now establish a matching theorem which gives precise meaning to these qualitative remarks. We shall discover that the operation on γ_1 should be one of a class of smoothing operators, while the operation on γ_2 should be one of a class of aperture-limiting operators. Then, by using the generalized Wiener-Khintchine relations (Lemma 3), which permit us to pass back and forth between the time and frequency domains, we develop alternative matching procedures which involve two smoothing or two aperture-limiting operations, one in the time domain and the other in the frequency domain.

3. The matching theorem

As a preliminary to the solution of the matching problem, we examine further the class of power schedules formed by multiplying the Gaussian function $\exp(-a\tau^2)$, $0 < a < \infty$, by an exponentially convex covariance. Using (2) we find that the canonical form of functions of this class is

$$f_a(\tau) = \exp(-a\tau^2) \int \exp(-\sigma\tau)p(\sigma)d\sigma \quad , \qquad 0 < a < \infty , \tag{8}$$

where $p(\sigma)$ is an integrable non-negative function (possibly containing delta function terms) such that the integral converges for all finite τ. Depending on the form of $p(\sigma)$, the positive function $f_a(\tau)$ may or may not be integrable.

Since, as already observed, we are interested only in integrable $f_a(\tau)$, we have to impose an appropriate restriction on $p(\sigma)$. Thus, we begin by indicating a relevant partition of the class of all integrable non-negative functions.

Definition. The integrable non-negative function $p(\sigma)$ is said to belong to the integrability class G_a, $0 < a < \infty$, symbolically $p(\sigma) \in G_a$, if the integral $\int \exp(\sigma^2/4a)p(\sigma)d\sigma$ is finite.

The following remarks concerning the classes G_a are easily verified:

1. $G_a \subset G_b$ if $a < b$.

2. There are $p(\sigma)$ in no G_a, e.g. $p(\sigma) = 1/(1+\sigma^2)$.

3. $\bigcap_{0 < a < \infty} G_a$ is non-empty and includes all $p(\sigma)$ vanishing outside a finite interval and other functions as well, e.g. $p(\sigma) = |\sigma|\exp(-\sigma^4)$.

4. If $c \neq 0$ and $p(\sigma) \in G_a$, then $p(c\sigma) \in G_{a'}$, where $a' = a/c^2$.

5. The integral $\int \exp(-\sigma\tau)p(\sigma)d\sigma$ is convergent for all finite τ if $p(\sigma)$ is in any G_a.

Lemma 1. The function $f_a(\tau)$ defined by (8) is integrable if and only if $p(\sigma) \in G_a$.

Proof. Since $\exp(-a\tau^2-\sigma\tau)$ is non-negative, by Fubini's theorem for non-negative functions we can use the existence of either iterated integral to infer the existence both of the other iterated integral and of the double integral. Thus suppose that $f_a(\tau)$ is integrable. Then

$$\int f_a(\tau)d\tau = \int \exp(-a\tau^2)d\tau \int \exp(-\sigma\tau)p(\sigma)d\sigma =$$

$$= \int p(\sigma)d\sigma \int \exp(-a\tau^2-\sigma\tau)d\tau = \sqrt{\pi/a} \int \exp(\sigma^2/4a)p(\sigma)d\sigma ,$$

so that $p(\sigma) \in G_a$. The converse follows by a detailed reversal of steps.

Lemma 2. If the function $f_a(\tau)$ defined by (8) is integrable, then it can be written as the convolution

(9) $$f_a(\tau) = \sqrt{a/\pi} \int \exp\left[-a(\tau-\sigma)^2\right] q(\sigma)\,d\sigma \quad , \quad 0 < a < \infty$$

of the Gaussian kernel $\sqrt{a/\pi}\,\exp(-a\tau^2)$ with the integrable non-negative function

(10) $$q(\sigma) = 2\sqrt{a\pi}\,\exp(a\sigma^2)p(-2a\sigma) \ .$$

Conversely, if the function $f_a(\tau)$ can be written as a convolution of the form (9), where $q(\sigma)$ is an integrable non-negative function, then $f_a(\tau)$ is integrable and can be written in the form (8) where

(10') $$p(\sigma) = \frac{1}{2\sqrt{a\pi}}\,\exp(-\sigma^2/4a)q(-\sigma/2a)$$

is an integrable non-negative function in the class G_a^*.

Proof. The converse follows at once by writing

$$\begin{aligned} f_a(\tau) &= \sqrt{a/\pi} \int \exp\left[-a(\tau-\sigma)^2\right] q(\sigma)\,d\sigma \\ &= \sqrt{a/\pi}\,\exp\,(-a\tau^2) \int \exp(2a\sigma\tau - a\sigma^2) q(\sigma)\,d\sigma \\ &= \frac{1}{2\sqrt{a\pi}}\,\exp(-a\tau^2) \int \exp(-\sigma\tau - \sigma^2/4a) q(-\sigma/2a)\,d\sigma \\ &= \exp(-a\tau^2) \int \exp(-\sigma\tau) p(\sigma)\,d\sigma \quad , \end{aligned}$$

where $p(\sigma)$ is given by (10') and is obviously in G_a. The integrability of $f_a(\tau)$ follows by Lemma 1.

The device used to prove the direct assertion is to take the Fourier transform of $f_a(\tau)$, regroup terms and then take the inverse Fourier transform. To justify freely exchanging orders of integration, we note that as in the proof of Lemma 1 the assumed integrability of $f_a(\tau)$ implies the integrability (in σ and τ) of $p(\sigma)\exp(-a\tau^2 - \sigma\tau)$, so that $p(\sigma)\exp(-a\tau^2 - \sigma\tau - i\omega\tau)$ is absolutely integrable, and the usual form of Fubinis theorem applies.

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* Of course, (10) and (10') are just two forms of the same relation.

Thus we write

$$2\pi F_a(\omega) = \int \exp(-i\omega\tau) f_a(\tau)\,d\tau = \int p(\sigma)\,d\sigma \int \exp(-a\tau^2 - \sigma\tau - i\omega\tau)\,d\tau$$

$$= \frac{1}{\sqrt{a}} \int \exp\left[\frac{1}{4a}(\sigma+i\omega)^2\right] p(\sigma)\,d\sigma \int_{-\infty+\frac{1}{2\sqrt{a}}(\sigma+i\omega)}^{\infty+\frac{1}{2\sqrt{a}}(\sigma+i\omega)} \exp(-u^2)\,du$$

$$= \sqrt{\pi/a}\, \exp(-\omega^2/4a) \int \exp(\sigma^2/4a)\exp(i\sigma\omega/2a)p(\sigma)\,d\sigma$$

$$= 2\sqrt{a\pi}\, \exp(-\omega^2/4a) \int \exp(a\sigma^2 - i\sigma\omega)p(-2a\sigma)\,d\sigma$$

$$= \exp(-\omega^2/4a) \int \exp(-i\sigma\omega)q(\sigma)\,d\sigma \quad ,$$

where $q(\sigma)$ is given by (10). The function $q(\sigma)$ is obviously non-negative and moreover is integrable since

$$q(\sigma)\,d\sigma = 2\sqrt{a\pi} \int \exp(a\sigma^2)p(-2a\sigma)\,d\sigma = \sqrt{\pi/a} \int \exp(\sigma^2/4a)p(\sigma)\,d\sigma \quad ,$$

where by Lemma 1 the existence of the last integral is implied by the integrability of $f_a(\tau)$. We now recover $f_a(\tau)$ in a new form by taking the inverse Fourier transform of $F_a(\omega)$ and using the convolution theorem for Fourier transforms. Thus we have finally

$$f_a(\tau) = \int \exp(i\omega\tau)F_a(\omega)\,d\omega = \sqrt{a/\pi} \int \exp\left[-a(\tau-\sigma)^2\right] q(\sigma)\,d\sigma \quad ,$$

the desired result (10). Finally we note that putting $\omega = 0$ in (11) gives $\int f_a(\tau)\,d\tau = \int q(\sigma)\,d\sigma$, so that convolving $q(\sigma)$ with the normalized kernel $\sqrt{a/\pi}\, \exp(-a\tau^2)$ preserves its integral.

The convolution transform (9) for $q(\sigma)$ in a variety of function classes has been extensively studied in the literature under the heading of the Weierstrass transform (see [5], ch.8); the kind of results stated in Lemmas 1 and 2 can certainly be assumed to belong to the folklore of that subject.

The next result is substantially the same as that derived in [1] under the name of the generalized Wiener-Khintchine relations:

Lemma 3. A necessary and sufficient condition for $x(t)$ to be a locally stationary random process is that the two-dimensional spectral density defined by

$$(12)\qquad \Psi(\omega,\omega') = \frac{1}{(2\pi)^2}\iint \exp\left[-i(\omega t-\omega' t')\right]\,\Gamma(t,t')\,dt\,dt'$$

with inversion

$$(13)\qquad \Gamma(t,t') = \iint \exp\left[i(\omega t-\omega' t')\right]\Psi(\omega,\omega')\,d\omega\,d\omega'$$

be a locally stationary covariance. In particular

$$(14)\qquad \Psi(\omega,\omega') = \Psi_1\left[(\omega+\omega')/2\right]\,\Psi_2(\omega-\omega')$$

where

$$(15)\qquad \Psi_1(\omega) = \frac{1}{2\pi}\int \exp(-i\omega\tau)\,\Gamma_2(\tau)\,d\tau\ ,\quad \Psi_2(\omega) = \frac{1}{2\pi}\int \exp(-i\omega\tau)\,\Gamma_1(\tau)\,d\tau\ ,$$

$$(16)\qquad \Gamma_1(\tau) = \int \exp(i\omega\tau)\,\Psi_2(\omega)\,d\omega\ ,\quad \Gamma_2(\tau) = \int \exp(i\omega\tau)\,\Psi_1(\omega)\,d\omega\ ,$$

i.e. Γ_1, Ψ_2 and Γ_2, Ψ_1 are Fourier transform pairs.

Proof. The form (14) and the relations (15) follow immediately from (12) if we make $\Gamma(t,t')$ locally stationary, change to new variables $(t+t')/2$ and $t-t'$, and use the identity $\omega t-\omega' t' = \frac{1}{2}\left[(\omega+\omega')(t-t')+(\omega-\omega')(t+t')\right]$. Similarly, if we use the same identity, the form (1) and the relations (16) follow from (13) and (14) after changing to new variables $(\omega+\omega')/2$ and $\omega-\omega'$. The assumed absolute integrability of $\Psi_2(\omega)$ (see Introduction) is used to justify the inversions for $\Gamma(t,t')$ and $\Gamma_1(\tau)$.

The functions Γ_1, Γ_2, Ψ_1 and Ψ_2 are, respectively, the power schedule, the temporal autocorrelation function, the power spectral density and

the spectral autocorrelation function of the process x(t). It follows from Lemma 3 that a locally stationary random process can be regarded as defined by any of the four pairs Γ_1 and Γ_2, Ψ_1 and Ψ_2, Γ_1 and Ψ_1, Γ_2 and Ψ_2. Thus, we shall call any of these four pairs components of the locally stationary process x(t).

Lemma 3 gives the harmonic analysis of the covariances $\Gamma(t,t')$ and $\Psi(\omega,\omega')$. In the Appendix we discuss the harmonic analysis of the process x(t) and of the locally stationary process $X(\omega)$ which has $\Psi(\omega,\omega')$ as its covariance.

We are now in a position to solve the matching problem proposed above:

Matching theorem. Let $\gamma_1(\tau)$ and $\psi_1(\omega)$ be any two integrable non-negative functions, and let $\gamma_2(\tau)$ and $\psi_2(\omega)$ be any two continuous autocorrelation functions. Then if new non-negative functions $\Gamma_1(\tau)$ and $\Psi_1(\omega)$ and new autocorrelation functions $\Gamma_2(\tau)$ and $\Psi_2(\omega)$ are constructed in any of the following four complementary ways, each of the four pairs Γ_1 and Γ_2, Ψ_1 and Ψ_2, Γ_1 and Ψ_1, Γ_2 and Ψ_2 are components of a (in general different) locally stationary random process. The four ways are:

1. Convolve $\gamma_1(\tau)$ with any Gaussian kernel $\sqrt{a/\pi}\,\exp(-a\tau^2)$, $0 < a < \infty$, to obtain $\Gamma_1(\tau)$; multiply $\gamma_2(\tau)$ by $\exp(-a\tau^2/4)$ to obtain $\Gamma_2(\tau)$;

2. Convolve $\psi_1(\omega)$ with $\sqrt{a/\pi}\,\exp(-a\omega^2)$ to obtain $\Psi_1(\omega)$; multiply $\psi_2(\omega)$ by $\exp(-a\omega^2/4)$ to obtain $\Psi_2(\omega)$;

3. Convolve $\gamma_1(\tau)$ with $\sqrt{a/\pi}\,\exp(-a\tau^2)$ to obtain $\Gamma_1(\tau)$; convolve $\psi_1(\omega)$ with $\frac{1}{\sqrt{a\pi}}\exp(-\omega^2/a)$ to obtain $\Psi_1(\omega)$;

4. Multiply $\gamma_2(\tau)$ by $\exp(-a\tau^2/4)$ to obtain $\Gamma_2(\tau)$; multiply $\psi_2(\omega)$ by $\exp(-\omega^2/4a)$ to obtain $\Psi_2(\omega)$.

Proof. By Lemma 2, given any integrable non-negative function $\gamma_1(\sigma)$, we can associate with it an integrable non-negative function $p(\sigma) \in G_a$ such that

$$g_a(\tau) = \exp(-a\tau^2) \int \exp(-\sigma\tau)p(\sigma)d\sigma = \sqrt{a/\pi} \int \exp\left[-a(\tau-\sigma)^2\right]\gamma_1(\sigma)d\sigma ,$$

where $g_a(\tau)$ is integrable and obviously continuous and non-negative. Then, given any continuous autocorrelation $\gamma_2(\tau)$, we see that the quantity

$$g_a\left[(t+t')/2\right]\exp\left[-\frac{a}{4}(t-t')^2\right]\gamma_2(t-t')$$

is a locally stationary covariance satisfying the continuity and integrability conditions. This follows by exactly the same reasoning as given in the Introduction. The rest of the proof follows by repeated use of the relations (15) and (16) and the convolution theorem for Fourier transforms. The class of locally stationary processes generated by the matching theorem is, of course, just the class Λ defined in the Introduction.

It is interesting to note that the smoothing operation called for by the matching theorem, i.e. convolution with a Gaussian kernel, is one of the so-called variation diminishing transforms of I.J. Schoenberg (See [5], ch.4). These transforms do not increase the number of zero-crossings of any absolutely integrable function to which they are applied, and therefore may be regarded as particularly effective smoothers.

4. Conclusion

We have seen how a pair of functions proposed as components of a locally stationary random process can be made to match each other by suitable smoothing and aperture-limiting operations. We find that the wider the proposed autocorrelation function, the more the proposed power schedule must be smoothed if the two functions are to match. Or, from a slightly different point of view, the smoother the power schedule, the less the aperture of the autocorrelation function has to be limited. The variety of different ways of describing the matching procedure is a natural consequence of the complementarity of smoothing a function and aperture-limiting its Fourier transform and vice versa.

Appendix

In the case where $x(t)$ is a locally stationary random process, the harmonic analysis of $x(t)$ itself is no more difficult to carry out than the harmonic analysis of its covariance. To see this, we observe first that the sample functions of $x(t)$ are square-integrable with probability one. This follows immediately from the integrability of $\Gamma(t,t)$, since

$$E\int |x(t)|^2 dt = \int E|x(t)|^2 dt = \int \Gamma(t,t)dt < \infty .$$

Thus, by Plancherel's theorem, with probability one the L^2 Fourier transform

$$X(\omega) = \frac{1}{2\pi}\int \exp(-i\omega t)x(t)dt \tag{17}$$

exists and is itself square-integrable.* In fact, with probability one we have $2\pi \int |X(\omega)|^2 d\omega = \int |x(t)|^2 dt$, and the inversion formula

$$x(t) = \int \exp(i\omega t)X(\omega)d\omega \tag{18}$$

gives back the original sample functions of $x(t)$, except perhaps for a set of t measure zero. (17) may be regarded as the defining equation of a new second order random process. It is easily seen that the covariance of $X(\omega)$ is just the two-dimensional spectral density $\Psi(\omega,\omega')$; in particular, it follows that $X(\omega)$ is itself a locally stationary process. The covariance of (18) is of course given by (13).

Thus we see that, unlike the case of globally stationary processes, the harmonic analysis of locally stationary processes can be achieved by taking Fourier transforms of the sample functions individually. The reason why the harmonic analysis of globally stationary processes is more complicated is not hard to find: In this case the quantity $\Gamma(t,t)$ is a positive constant

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* The integrals in (17) and (18) are to be interpreted as l.i.m. $\int_{-A}^{A}$ as $A \to \infty$.

so that with probability one the sample functions of $x(t)$ have no Fourier transforms. In particular (18) can no longer be written, but must be replaced by

$$x(t) = \int \exp(i\omega t)\,dX(\omega)$$

where now the integral cannot be interpreted as an ensemble of integrals over separate sample functions, which would be meaningless, but as a stochastic integral (see [3], p. 472). It is clear that this complication arises from the unphysical assumption that with probability one the sample functions of $x(t)$ have infinite energy. In the author's opinion, the fact that the harmonic analysis of locally stationary processes is so straightforward may well enhance their utility as a tool for solving physical problems.

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